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On Stability Criteria of the Kelley-McShane Linearized Theory of Yawing Motion

C. H. MURPHY

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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ON STABILITY CRITERIA OF THE KELLEY-McSHANE LINEARIZED THEORY OF YAWING MOTION

C. H. MURPHY

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ABERDEEN PROVING GROUND, MARYLAND

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ABSTRACT

A slightly more accurate dynamic stability criterion arising from a more careful treatment of a perturbation term and incorporating a revised treatment of the gravity term is derived. The new criterion has the property of reducing to the classical ballistic stability requirement $s \geq 1$ when only K_M is retained while the older criteria became indeterminate for this case. The criterion is then generalized by the insertion of the requirement of a minimum level of damping which is not necessarily zero. Comparisons are made with the other criteria and definition of the terms "dynamic stability", "gyroscopic stability" and "static stability" are stated and discussed. The effect of spin on dynamic stability is indicated and the effect of the location of the center of mass is discussed in some detail. A relation for an optimum center of mass location is obtained. The equations of yawing motion are derived in an appendix and solved in the body of the report. The Kelley-McShane force system is described and the resulting dependence of the aerodynamic coefficients on center of mass position is derived in a second appendix. A third appendix converts the expressions used in the body of the report from ballistic to aerodynamic nomenclature.

INTRODUCTION

The Kelley-McShane linearized equations of yawing motion are simple linear second order equations which possess only one complication. This is caused by the appearance of ν, the spin in radians per caliber, in certain of the coefficients. Since this spin is a slowly varying function of position, one is forced to consider the effect of slowly varying coefficients. In order to handle this Kelley and McShane [5] [6][1] introduced a perturbation term. The effect of this term on the stability has not been completely discussed. In fact most of it is omitted from the stability criterion of [5] and some of it is omitted from that of [6][2]. The clarification of this discrepancy and the resulting derivation of a slightly more accurate stability criterion is one of the aims of this report. This criterion will have the further advantage of incorporating the correct contribution of gravity as is indicated in [24].

It is found that most considerations of stability are complicated by unnecessarily complex subdivisions of stability which arise from the sign of the moment coefficient and the presence or absence of spin. It is interesting to note that the Kelley-McShane dynamic stability criterion actually becomes indeterminate when the effect of all aerodynamic coefficients other than K_M is neglected. It will be shown in the body of this report that the Kelley-McShane criterion can be revised so that it will reduce to the expected result of the "classical" ballistic stability requirements for this special case and is independent of any restriction on K_M or ν.

The dynamic stability criterion can then be generalized by the introduction of the concept of a minimum level of stability. The classical stability definition is founded on the requirement that the two damping coefficients α_1 and α_2 be non-negative. The generalized requirement is derived from the requirement that α_1 and α_2 be not less than an assigned number α which can be described as the minimum permissible level of damping.

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1. Numbers in brackets refer to specific publications listed in the references on page [28].
2. To be precise in [5], s_1, s_2 and s_3 which appear in the stability criterion each receive a $J_D - J_g$ from the perturbation term and in [6] they receive a $J_D - J_g - \frac{md^2}{A} J_A$ from the perturbation term. J_g is defined to be the " $- gdu^{-2}$ sin θ" of [5] and [6].

A further confusion in stability studies is found in the many types of stability which are discussed. The most common of these are "dynamic stability", "static stability" and "gyroscopic stability". Precise definitions of these terms and a discussion of their interrelationship would be of some value.

The effects of spin and especially of center of mass location on dynamic stability of a "statically unstable" missile are quite interesting and are discussed in some detail. At this point a dynamic stability factor $\bar{s}$ is introduced.

The purposes of this report are then:

1. To derive a dynamic stability criterion which contains the full contribution of the perturbation term and has the correct contribution of gravity.

2. To state this stability criterion in a fashion which is independent of the sign of K_M and the presence or absence of nonzero spin and which is equivalent to the "classical" ballistic stability when only K_M is retained.

3. To generalize the dynamic stability criterion by introduction of a minimum permissible level of damping.

4. To define the terms "dynamic stability", "static stability" and "gyroscopic stability" with respect to this criterion so that all interrelationships are clear.

5. To discuss the effect of spin and center of mass location on dynamic stability.

SOLUTIONS OF THE EQUATIONS OF YAWING MOTION

The equations of yawing motion, which are derived in the appendix with attention to proper treatment of the gravity terms, may be stated as:

$$(1) \quad \lambda'' + \left[H + J_g - i\bar{\nu}\right]\lambda' + \left[-M - i\bar{\nu}T\right]\lambda = G$$

$$(2) \quad \bar{\nu}' = \left(D - J_g\right)\bar{\nu}$$

where $\lambda = \lambda_2 + i\lambda_3$ complex yaw

$$H = J_L - J_D + \frac{md^2}{B} \quad J_H$$

$$J_g = \frac{g_1 d}{u_1^2} \doteq -\frac{gd \sin\theta}{u^2}$$

1. This solution is essentially that described in the original Kelley-McShane report [5].

θ = inclination of the trajectory

$$\bar{\nu} = \frac{A}{B}\nu = \frac{A}{B}\frac{\omega_1 d}{u_1}$$

$$M = \frac{md^2}{B} J_M + \nu\bar{\nu} J_F \pm \frac{md^2}{B} J_M$$

$$T = J_L - \frac{md^2}{A} J_T$$

$$G = \gamma' - \left[\left(J_D - J_g - \frac{md^2}{B} J_H\right) + i\bar{\nu}\right]\gamma$$

$$D = J_D - \frac{md^2}{A} J_A$$

m = mass of the missile

d = diameter of the missile

B = transverse moment of inertia

A = axial moment of inertia

(g_1, g_2, g_3) = acceleration due to gravity vector where the one axis is along missile's axis

u_1 = axial component of velocity

ω_1 = axial component of angular velocity

$$\gamma = \frac{\left[(g_2 - g_1\lambda_2) + i(g_3 - g_1\lambda_3)\right]d}{u_1^2}$$

$$J_i = \frac{\rho d^3}{m} K_i$$

ρ = density of air

K_i are the ballistic coefficients and primes refer to differentiation with respect to the axial arc length along the trajectory in calibers.

If the homogeneous equation is first considered, the problem reduces to the solution of a second order linear differential equation with slowly varying coefficients. Equation (1) may be simplified by the substitution $\lambda = q \exp 1/2\int_0^p -(H + J_g - i\bar{\nu})\, dp$ which eliminates the first derivative term.

If the [illegible] considerations are employed, equation (1) can be written [illegible] $q'' - r^2 q = 0$

where $r^2 = 1/4 \left[(4M - \bar{v}^2) + (H + J_g)^2 + 2 i\bar{v} (2T - H - D)\right]$

Equation (3) can be further simplified by use of a second transformation $w = (\ln q)'$

(4) $w' + w^2 - r^2 = 0$

Clearly if r were a constant, a pair of solutions of (4) is $w = \pm r$. Closer examination would reveal that these are singular solutions of the non-linear equation (4). The solutions of the linear (1) which correspond to these are linearly independent and hence a linear combination of them would be the general solution of that equation. Here we have the interesting occurence of the singular solution of one equation forming the general solution of an equation derived from it. We now assume that for non-constant r one solution of (4) is of the form $r - 1/2\,\varepsilon$ where ε is small.[1]

(5) $r' - 1/2\,\varepsilon' - \varepsilon r + 1/4\,\varepsilon^2 = 0$

or if $-1/2\,\varepsilon' + 1/4\,\varepsilon^2$ is small in comparison with r' and $r \neq 0$

(6) $\varepsilon = \frac{r'}{r}$

But by use of (2) we have[2]

(7) $\frac{r'}{r} = \frac{(4r^2)'}{2(4r^2)} = (D - J_g)\,\bar{v}\left[\frac{\bar{v} - i\,(2T - H - D)}{\bar{m} - 2i\bar{v}\,(2T - H - D)}\right]$;

$\bar{m} = \bar{v}^2 - 4M - (H + J_g)^2$

---

1. The -1/2 is introduced here in order to obtain a slight simplification in the final form of the solution. This perturbation method is called the W.K.B. method (Wentzel, Kramers, Brillouin) and is described by H. Jeffreys in Proc. of the London Math. Soc. (2) vol 23 (1923) p. 428.

2. $\bar{v}^2 - 4M$ is usually written as $\bar{v}^2 \sigma^2$ where $\sigma^2 = 1 - \frac{1}{s}$ and s, the ballistic stability factor, is $\frac{\bar{v}^2}{4M}$. The use of a squared symbol implies that $\bar{v}^2 - 4M$ must be positive and the definition of σ^2 has the further disadvantage of becoming infinite for zero spin. In order to avoid these handicaps $\bar{m}$ is introduced.

If a second solution of the form $-r - 1/2\mathcal{E}^*$ is considered, it follows directly that $\mathcal{E}^* = \mathcal{E}$ and hence two solutions of (4) are $r - 1/2\mathcal{E}$ and $-r - 1/2\mathcal{E}$. From this the general solution of (1) is obtained as

$$(8)\ \lambda = K_1 \exp 1/2 \int^p \left\{-H - J_g + i\bar{v} - \mathcal{E} + \left[-\bar{m} + 2i\bar{v}\,(2T - H - D\right]^{1/2}\right\} dp$$

$$+ K_2 \exp 1/2 \int^p \left\{-H - J_g + i\bar{v} - \mathcal{E} - \left[-\bar{m} + 2i\bar{v}\,(2T - H - D)\right]^{1/2}\right\} dp$$

If we write $\mathcal{E}$ in the form $(D - J_g)(\mathcal{E}_1 + i\,\mathcal{E}_2)$, it is easily seen that only $\mathcal{E}_1$ should be considered. This follows from the fact that $(D - J_g)\mathcal{E}_2$ is small when compared with $\bar{v}$ and certainly can be ignored in equation (8).

Now (9) $\mathcal{E}_1 = \dfrac{\bar{v}^2\left[\bar{m} + 2\,(2T - H - D)\right]^2}{\bar{m}^2 + 4\,\bar{v}^2\,(2T - H - D)^2}$; $\bar{m} = \bar{v}^2 - 4\dfrac{md^2}{B}\,J_M - (H + J_g)^2$

For most missiles $|\bar{m}| > |2T-H-D|$ and $\bar{v} < 1/20$ and we have the usual approximation $\mathcal{E}_1 \sim \dfrac{\bar{v}^2}{\bar{m}}$.

Assuming that H, T, and D are constants it can be shown that for a flat trajectory ($J_g = 0$) (8) can be approximated quite accurately by

$$(10)\ \lambda = K_1 \exp(-\alpha_1 + i\phi_1')\,p + K_2 \exp(-\alpha_2 + i\phi_2')\,p$$

where ϕ_1' and ϕ_2' are linear functions of p

α_1 and α_2 are constant functions of p

$$\bar{v} = \phi_1' + \phi_2'$$

$$H = (\alpha_1 + \alpha_2) - D\,\mathcal{E}_1$$

$$M = \phi_1' \cdot \phi_2' - \alpha_1\alpha_2 + \frac{\mathcal{E}_1}{2}\,D(\alpha_1 + \alpha_2 - \frac{\mathcal{E}_1}{2}\,D)$$

$$T = -1/2\left[(\alpha_1 - \alpha_2)\,\frac{\phi_1' - \phi_2'}{\phi_1' + \phi_2'} - H - D\right]$$

$$D = \frac{\phi_1'' + \phi_2''}{\phi_1' + \phi_2'} \quad \text{(see note 1)}$$

1. The first four relations following (10) are derived for the case of constant ϕ_i' 's in [25]. It can be easily shown that they are true for the case of linear ϕ_i' 's. The fifth relation comes from the first by the use of (2). The relation for M differs from that in [25] due to the retention of $(H + J_g)^2$ in $\bar{m}$.

Although the solution of the inhomogeneous equation is not required for the stability discussion, we will state it here for completeness. This solution adds an almost constant "yaw of repose" λ_R to the general solution. The relation defining λ_R is derived in [24] and may be stated in the form

$$(11)\quad \lambda_R = \frac{gd \cos \theta}{M u^2}\left[-\bar{v} + i\left(2J_g - J_D - \frac{md^2}{B} J_H + \bar{v}^2 \frac{T}{H}\right)\right]\left[1 + \left(\frac{\bar{v}T}{H}\right)^2\right]^{-1}$$

where θ is the inclination of the trajectory.

STABILITY CRITERION

Dynamic Stability is defined by the statement that a missile is dynamically stable if the yaw described by the solution of the homogeneous equation does not increase.[1] Referring to (8) it can be seen that this requires that $H + J_g + (D - J_g)\mathcal{E}_1$ be greater than or equal to the real part of the square root and be non-negative.[2] This is equivalent to the relation:

$$(12)\quad H + \bar{\mathcal{E}} \geqq \left| R\left\{\left[-\bar{m} + 2i\bar{v}\,(2T - H - D)\right]^{1/2}\right\}\right|$$

where $\bar{\mathcal{E}} = J_g + (D - J_g)\mathcal{E}_1 = J_g\,(1 - \mathcal{E}_1) + D\,\mathcal{E}_1$.

If the square root is replaced by $\sqrt{a + ib}$ where $a = -\bar{m}$, $b = 2\bar{v}\,(2T - H - D)$, $a + ib = \sqrt{(a^2 + b^2)}(\cos C + i \sin C)$ and $\tan C = \frac{b}{a}$,

De Moivre's Theorem states that:

$$(13)\quad R\left(\sqrt{a + ib}\right) = R\left(\sqrt[4]{a^2 + b^2}\;\left(\cos \frac{C}{2} + i \sin \frac{C}{2}\right)\right)$$

$$= \sqrt[4]{a^2 + b^2}\;\cos \frac{C}{2}$$

Substituting this in (12) and using the half angle formula for $\cos \frac{C}{2}$ gives the following equations:

$$(14)\quad H + \bar{\mathcal{E}} \gtreqless \sqrt[4]{a^2 + b^2}\;\sqrt{\frac{1 + \cos C}{2}}$$

$$\text{or}\quad H + \bar{\mathcal{E}} \gtreqless \sqrt[4]{a^2 + b^2}\;\sqrt{\frac{1 + a/\sqrt{a^2 + b^2}}{2}}$$

$$\text{or}\quad H + \bar{\mathcal{E}} \gtreqless \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}}$$

---

1. Since the initial conditions affect only the homogeneous part of the solution, this requirement is equivalent to the restriction that the size of yaw caused by the initial conditions will not increase throughout the trajectory. Although the yaw of repose may increase, it is unaffected by the random initial conditions and its effect may be computed.
2. Note that $\mathcal{E}$ is here replaced by $(D - J_g)\mathcal{E}_1$.

Squaring this we have

$$(15)\quad 2\left[H+\bar{\varepsilon}\right]^2 \geq \sqrt{a^2+b^2}+a,\quad H+\bar{\varepsilon} \geq 0$$

The second inequality is required in order that (15) be equivalent to (14). Regrouping and squaring again we have

$$(16)\quad \left[2(H+\bar{\varepsilon})^2-a\right]^2 \geq a^2+b^2,\quad H+\bar{\varepsilon} = 0$$

Unfortunately (16) is not precisely equivalent to (15) since the first inequality of (16) implies either

$$(17)\quad 2(H+\bar{\varepsilon})^2-a \geq \sqrt{a^2+b^2} \text{ or } a-2(H+\bar{\varepsilon})^2 \geq \sqrt{a^2+b^2}$$

In order that (16) be equivalent to (15), it must be possible to reject the second inequality of (17). But this is only possible if $H+\bar{\varepsilon} \neq 0$. If $H+\bar{\varepsilon}=0$, (15) requires that $b=0$ and that a be non-positive. With this in mind we can revise (16) to the following form which is equivalent to (15).

$$(16')\quad H+\bar{\varepsilon}>0 \qquad\qquad \text{or} \qquad\qquad H+\bar{\varepsilon}=0$$

$$\left[2(H+\bar{\varepsilon})^2-a\right]^2 \geq a^2+b^2 \qquad\qquad b=0$$

$$a \leq 0$$

If the inequalities are solved for $-a$, and a and b are replaced by their definitions,

$$(18)\quad H+\bar{\varepsilon}>0 \qquad\qquad \text{or} \qquad\qquad H+\bar{\varepsilon}=0$$

$$\bar{v}^2-4M \geq 0 \qquad\qquad \bar{v}(2T-H-D)=0$$

$$\bar{v}^2-4M \geq 0$$

$$\bar{v}^2-4M = \bar{v}^2\left[\frac{(2T-H-D)}{H+\bar{\varepsilon}}\right]^2-(H+\bar{\varepsilon})^2+(H+J_g)^2$$

These are the exact stability conditions. The usual size approximations can now be applied which state J^2 terms can be dropped[1] and the following theorem is obtained:

---

1. Note that for constant spin $D-J_g=0$ or zero spin $v=0$, these J^2 terms vanish exactly from (18). Hence the theorem is not an approximation of (18) for this case.

Theorem In the linearized theory a missile possesses dynamic stability if either of the following sets of relations is satisfied.

(19a) $H + \bar{\mathcal{E}} > 0$

$$\bar{v}^2 - 4M \gtrless \bar{v}^2 \left(\frac{2T-H-D}{H + \bar{\mathcal{E}}} \right)^2$$

(19b) $H + \bar{\mathcal{E}} = 0$

$$\bar{v}(2T-H-D) = 0$$

$$\bar{v}^2 - 4M \gtrless 0$$

where $\bar{\mathcal{E}} = J_g (1 - \mathcal{E}_1) + D\mathcal{E}_1$

$$\mathcal{E}_1 = \frac{\bar{v}^2 \left[(\bar{v}^2 - 4M) + 2(2T-H-D)^2 \right]}{(\bar{v}^2 - 4M)^2 + 4\bar{v}^2 (2T-H-D)^2} \sim \frac{\bar{v}^2}{\bar{v}^2 - 4M}.$$

$$\bar{v}^2 - 4M > |2T-H-D|$$

Note. (19a) may be obtained somewhat more simply by the assumption that $\bar{m} = \bar{v}^2 - 4M > 0$ together with the approximation of a binomial expansion of the square root in (8). If this is done, the α_1 and ϕ_1' of (10) become

$$(20) \quad \alpha_{1,2} = 1/2 \left[H + \bar{\mathcal{E}} \mp \frac{\bar{v}}{\sqrt{\bar{m}}} (2T-H-D) \right] \sim 1/2 \left[(1 \pm \tfrac{1}{\sigma}) H \mp \tfrac{2}{\sigma} T + (\tfrac{1 \pm \sigma}{\sigma^2}) D \right]$$

$$\phi'_{1,2} = 1/2\, \bar{v}\, (1 \pm \sigma)$$

where $\sigma = \frac{\sqrt{\bar{m}}}{\bar{v}}$

$$\bar{\mathcal{E}} \sim \frac{D\bar{v}^2}{\bar{m}} = \frac{D}{\sigma^2}$$

Dynamic stability requires that $\alpha_{1,2} \gtrless 0$ and this implies that

$$H + \bar{\mathcal{E}} \gtrless \frac{|\bar{v}(2T-H-D)|}{\sqrt{\bar{m}}} \quad \text{or}$$

$$(21) \quad \sqrt{\bar{v}^2 - 4M} \gtrless \frac{|\bar{v}(2T-H-D)|}{H + \bar{\mathcal{E}}}$$. But this is equivalent to (19a).

Although this derivation is handicapped by the assumptions made above, with some modifications this reasoning can be made rigorous.

Since relations (19) require $\bar{v}^2 - 4M$ to be positive, $\mathcal{E}_1$ must be positive and considered as a function of spin it attains a minimum of one for infinite spin. If the slope, $D-J_{\bar{\mathcal{E}}}$, is positive[1] then the minimum value of $\mathcal{E}_1$ corresponds to the minimum $\bar{\mathcal{E}}$ and the following sufficient stability condition follows from (19a).

(22) $H + D > 0$

$$\bar{v}^2 - 4M \gtrless \bar{v}^2 \left[\frac{2T-H-D}{H + D}\right]^2$$

If $D = J_D - \frac{md^2}{A} J_A$ is approximated in $H + D$ by J_D, there results a set of conditions which are sufficient for stability over flat trajectories when $\mathcal{E}_1 < \frac{J_D}{D}$.

(23) $H + J_D > 0$

$$\bar{v}^2 - 4M \gtrless \bar{v}^2 \left[\frac{2T-H-D}{H + J_D}\right]^2$$

Equations (23) are precisely the stability criteria of [5] with the correct treatment of gravity,[2] and equations (22) are the criteria of [6]. Therefore if only flat trajectories are considered and $\mathcal{E}_1 < \frac{J_D}{D}$, the stability criteria of [5] and [6] are sufficient conditions for stability.[3] No indication of the second set of relations of (19) can be found in (22) or (23) and herein is the reason for the failure of the criteria of [5] or [6] to reduce to the classical gyroscopic stability criteria, $s \gtrless 1$.

---

1. This is satisfied over all of the upward branch and part of the downward branch of any trajectory and is true over the entire length of flat trajectories dealt with in spark range work. The minimum value of $\mathcal{E}_1$ is one and hence the minimum $\bar{\mathcal{E}}$ is D.
2. In [5] part of $\bar{\mathcal{E}}$ is integrated out and not considered in the stability analysis. From the remainder the correct treatment of gravity [25] produces (23) which is independent of gravity.
3. It must be emphasized that from a practical engineering standpoint (19), (22) and (23) are equivalent and, hence, (22) being the simplest, should be used.

Before going on to a discussion of various types of stability and the effect of spin and center of mass location on stability we will first obtain a generalized form of (19) which should be quite useful in design work. (19) was obtained from the requirement that $\alpha_{1,2} \gtrless 0$. If this were replaced by the requirement that $\alpha_{1,2} \gtrless \alpha$ where α is the minimum permissible damping, the criterion becomes much more flexible and valuable to the design ballistician or aerodynamicist.

This revision can be done quite easily by replacing (12) by

$$(12') \quad H + \bar{\mathcal{E}} - \left| R\left\{ -\bar{m} + 2i\bar{\nu} \quad (2T-H-D)^{1/2} \right\} \right| \gtrless \alpha$$

(12') can be reduced to (12) by writing $\bar{\mathcal{E}}$ for $\bar{\mathcal{E}} - \alpha$. This means that the generalized criterion can be obtained by inserting $\bar{\mathcal{E}} - \alpha$ for $\bar{\mathcal{E}}$ in (20).

Theorem In the linearized theory the damping rates of a missile are greater than α when either of the following sets of relationsis satisfied.[1]

$$(24a) \quad H + \bar{\mathcal{E}} - \alpha > 0$$

$$\bar{\nu}^2 - 4M \gtrless \bar{\nu}^2 \left[\frac{2T-H-D}{H + \bar{\mathcal{E}} - \alpha} \right]^2$$

$$(24b) \quad H + \bar{\mathcal{E}} - \alpha = 0$$

$$\bar{\nu}(2T-H-D) = 0$$

$$\bar{\nu}^2 - 4M \gtrless 0$$

DISCUSSION OF STABILITY

In addition to the concept of dynamic stability two other types of stability are usually considered: gyroscopic and static. Their definitions follow directly from the definition of dynamic stability for the case where only the effect of K_M is considered. For this case the second set of relations of (19) must be used and they reduce to

$$(25) \quad \bar{\nu}^2 - 4M \gtrless 0$$

1. Note that α can be negative although its magnitude is restricted to the order of a J term. This means that should a little instability be permitted the generalized criterion will still apply.

A missile is said to be gyroscopically stable if it satisfies inequality (25). For zero spin (25) becomes $M \leqq 0$ or $K_M \leqq 0$. A missile for which $K_M \leqq 0$ is said to be statically stable.[1] For statically unstable missiles $K_M > 0$ and inequality (25) can be written in the form:

$$(25') \quad \frac{\bar{\nu}^2}{4M} = \frac{A^2 \omega_1^2}{4B \rho d^3 u_1^2 K_M} \gtreqless 1$$

This is, of course, the classical ballistic stability condition

$s \gtreqless 1$ where $s = \dfrac{A^2 \omega_1^2}{4B \rho d^3 u_1^2 K_M}$. Gyroscopic stability for statically unstable missiles, therefore, reduces to the classical stability condition.

The logical connections between gyroscopic stability, static stability, and the basically fundamental dynamic stability can be derived from their respective definitions. They are shown in figure 1 and are listed below.

(1) If only K_M is considered for stability, gyroscopic stability is sufficient for dynamic stability.

(2) Gyroscopic stability is always necessary for dynamic stability.

(3) If spin is zero, static stability is necessary for gyroscopic stability.

(4) For any spin, static stability is sufficient for gyroscopic stability.

(5) If $H + \bar{\mathcal{E}} \gtreqless 0$ and $\nu = 0$, static stability is sufficient for dynamic stability.

(6) If $H + \bar{\mathcal{E}} \gtreqless 0$ and $2T - H - D = 0$, gyroscopic stability is sufficient for dynamic stability.[2]

1. Since K_M is defined so that it is positive when the center of pressure is in front of the center of mass, most bodies of revolution are statically unstable while most finned missiles are statically stable.
2. This relationship is quite interesting since it indicates a sort of optimum center of mass location i. e. such that $2T-H-D = 0$. This relationship is implied by R. Turetsky in [18].

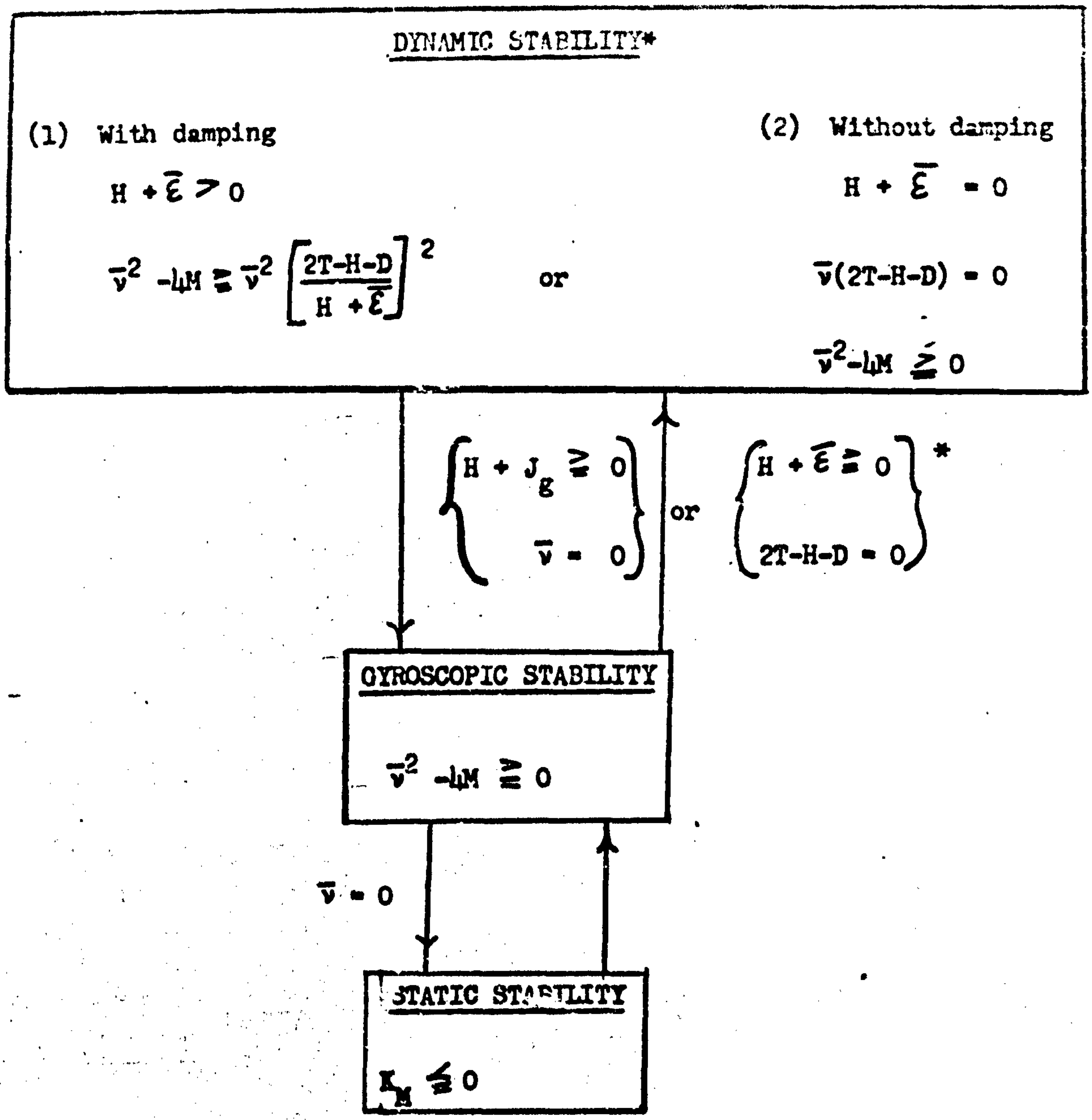

Fig. 1

$$* \quad \bar{\mathcal{E}} = J_g (1 - \mathcal{E}_1) + D\,\mathcal{E}_1$$

$$\mathcal{E}_1 = \frac{\bar{\nu}^2 \left[(\bar{\nu}^2 - 4M) + 2(2T-H-D)^2\right]}{(\bar{\nu}^2 - 4M)^2 + 4\bar{\nu}^2 (2T-H-D)^2} \sim \frac{\bar{\nu}^2}{\bar{\nu}^2 - 4M}$$

In order to discuss the particular effect of spin, we will use the slightly simpler sufficient condition (22) and will assume H + D to be always positive. (22) can be written

$$(26)\quad \bar{v}^2 \frac{T(H + D - T)}{(H + D)^2} \gtrless M \quad \text{or} \quad \bar{v}^2\, \bar{s}(2-\bar{s}) \gtrless 4M$$

Where $\bar{s} = \frac{2T}{H + D} = \frac{2(K_L - \bar{k}_1^2 K_T)}{K_L + \bar{k}_2^2 K_H - \bar{k}_1^2 K_A}$ (Dynamic Stability Factor)

$$k_1^{-2} = \frac{md^2}{A} \quad (k_1 \text{ is axial radius of gyration in calibers})$$

$$k_2^{-2} = \frac{md^2}{B} \quad (k_2 \text{ is transverse radius of gyration in calibers})$$

If $\bar{s}$ is either 0 or 2, the coefficient of v^2 is zero and for this case a statically unstable missile can not be made dynamically stable by spin while a statically stable missile is dynamically stable regardless of spin. Otherwise (26) can be solved for v^2 and two possibilities arise:

$$(27a)\quad \bar{s}(2 - \bar{s}) > 0; \qquad \bar{v}^2 \gtrless \frac{4M}{\bar{s}(2 - \bar{s})}$$

$$(27b)\quad \bar{s}(2 - \bar{s}) < 0; \qquad \bar{v}^2 \lesseqgtr \frac{4M}{\bar{s}(2 - \bar{s})}$$

From this we see that:

(1) For a statically stable missile either $\bar{s}(2 - \bar{s}) \gtrless 0$ and it is dynamically stable independent of spin or $\bar{s}(2 - \bar{s}) < 0$ and dynamic stability places an upper bound on spin by use of (27b).[1]

(2) For a statically unstable missile either $\bar{s}(2 - \bar{s}) > 0$ and dynamic stability places a lower bound for spin by use of (27a) or $\bar{s}(2 - \bar{s}) \lesseqgtr 0$ and the missile is dynamically unstable independent of spin.[2]

---

1. Dynamic data obtained from the simple finned configurations described in [17] show that this upper bound actually exists! At Mach number of 1.7 $\bar{s}$ is -7.2 and for this value of $\bar{s}$ the upper bound on spin is 4.2° per caliber. One round has been fired at a spin rate of 4.0° per caliber and it was found that the slow arm had practically no damping. At Mach No. of 1.3, $\bar{s}$ = - 30 and model was unstable.
2. This entire discussion of the effect of spin on stability is essentially that of Kelley-McShane in either [5] or [6]. The s_i's of Kelley-McShane may be related to the symbols of this report by $s_1 = H + \bar{\mathcal{E}}$, $s_2 = 2T + \bar{\mathcal{E}} - D$, and $s_3 = 2(H-T) + \bar{\mathcal{E}} + D$. The discussion is inserted here in order to give this report completeness. Here again the case of the airship must be excluded. (See Appendix B).

For the assumption made above (namely $H + \bar{\mathcal{E}} = H + D > 0$), the effect of spin on dynamic stability may be stated in a very concise manner: Theorem If $\bar{s}(2 - \bar{s})$ and M have the same sign, a missile is dynamically stable if $s = \frac{\bar{v}^2}{4M} \gtrless \frac{1}{\bar{s}(2 - \bar{s})}$.

Otherwise spin has no effect and statically stable missiles are dynamically stable while statically unstable missiles are dynamically unstable.

We will now consider the interesting question of the effect of center of mass location on the stability of a statically unstable missile. In order to do this we need to convert the generalized stability condition into a more useful and specialized form. Straightforward algebra and the approximation $\bar{\mathcal{E}} \doteq D$ can be employed to obtain the following result.

Theorem

The damping exponents α_1 and α_2 of the epicyclic yawing motion of a statically unstable missile are greater than or equal to an assigned value α if either of the following sets of relations are satisfied. (If $\alpha = 0$, the relation of the first set became precisely those for the dynamic stability given in equation (27a)).

(28a) $H + D - \alpha > 0$	or (29a) $H + D - \alpha = 0$
(28b) $\frac{\bar{v}^2}{4M} = s \gtrless \frac{1}{\bar{s}(2-\bar{s})}$	(29b) $2T - \alpha = 0$
(28c) $0 < \bar{s} < 2$	(29c) $s \gtrless 1$

where $\bar{s} = \bar{s}(\alpha) = \frac{2T - \alpha}{H + D - \alpha}$ (Generalized Dynamic Stability Factor)

Relations (29) are only of academic interest for designers and will be disregarded in the remainder of this report. The form of relations (28) has been greatly simplified by the introduction of $\bar{s}(\alpha)$ which will be called "the generalized dynamic stability factor". Whenever $\bar{s}$ appears without a reference to a value of α, α will be taken to be zero and $\bar{s}$ is then called "the dynamic stability factor". If we multiply the numerator and denominator of $\bar{s}(\alpha)$ by $\frac{m}{\rho d^3}$ it reduces to the following simple form:

$$\bar{s}(\alpha^*) = \frac{2(K_L - k_1^{-2} K_T) - \alpha^*}{K_L + k_2^{-2} K_H - k_1^{-2} K_A - \alpha^*}$$

where $\alpha^* = \frac{m}{\rho d^3} \alpha$

The equation $a = \frac{1}{\bar{s}(2 - \bar{s})}$ is plotted in Figure 2. The curve is a fairly flat one with asymptotes at zero and two. The modification of the classical gyroscopic stability requirement by the dynamic stability factor is significant only when $\bar{s}$ is outside the interval from 3/4 to 5/4. The center of this interval which is $\bar{s} = 1$ corresponds to the optimum center of mass position mentioned before (2T-H-D = 0) and $\bar{s}$ is independent of α here. This means that if the gyroscopic stability factor is unity for this center of mass position, an increase in spin which in turn increases "s" will not improve the dynamic stability. As a matter of fact equation (20) shows that both arms damp at the same rate which is independent of spin and that this rate is $H + \mathcal{E} \sim H + D$. Equation (20) also reveals the interesting property that for $\bar{s} < 1$ the slow arm has the smaller damping rate while for $\bar{s} > 1$ the reverse is true.[1] This is important since α of the criterion is the damping rate of the arm possessing the smaller damping rate.

We now return to the center of mass effect on stability. From appendix A we have the result that K_A and K_L are independent of center of mass location while K_T and K_H are linear and quadratic functions respectively of its location.

(30) $T = T_0 + T_1 q$

$$H + D = H_0 + H_1 q + H_2 q^2$$

where $T_0 = J_L - k_1^{-2} J_T$

$$T_1 = k_1^{-2} J_F$$

$$H_0 = J_L + k_2^{-2} J_H - k_1^{-2} J_A$$

$$H_1 = -k_2^{-2} (J_S + J_M)$$

$$H_2 = k_2^{-2} J_N$$

---

1. By slow or fast arm is meant the arm with the smaller or larger turning rate ϕ_1'. The slow arm is sometimes called the precessional arm while the fast arm is called the nutational arm. Since in the classical theory of the top the precessional motion is usually considered to be the motion of the plane of yaw and the nutational motion is the variation of the magnitude of yaw from maximum to minimum, this nomenclature is conflicting. (It can be shown that nutation rate is actually $\phi_1' - \phi_2'$ while the precessional rate is variable unless K_1 or K_2 is zero)

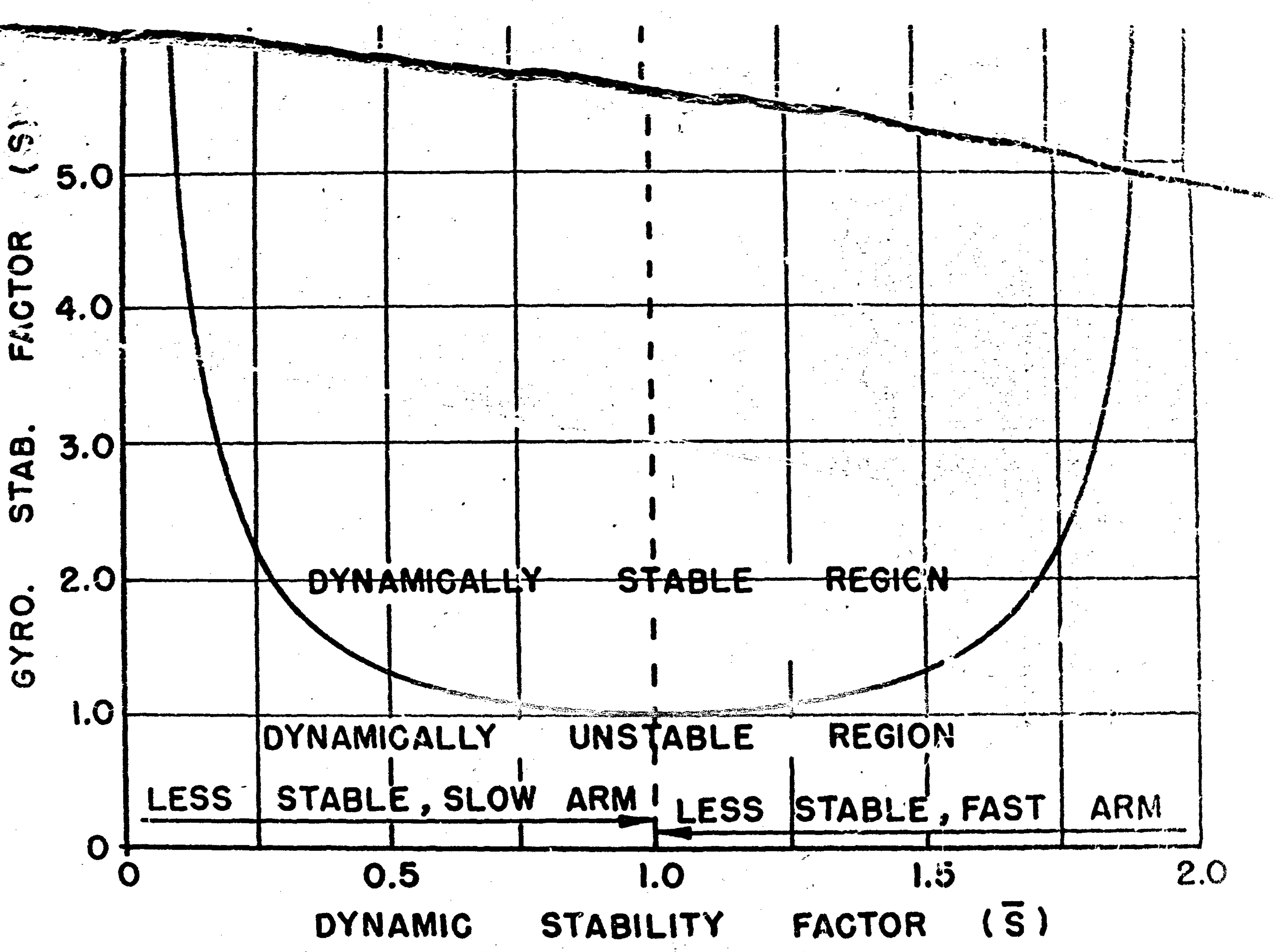
GYRO. STAB. FACTOR (S)
5.0
4.0
3.0
2.0
1.0
0
0
0.5
1.0
1.5
2.0
DYNAMIC STABILITY FACTOR ($\bar{S}$)
DYNAMICALLY STABLE REGION
DYNAMICALLY UNSTABLE REGION
LESS STABLE, SLOW ARM
LESS STABLE, FAST ARM

FIG. 2

q is the shift of c.m. in calibers and is bounded to a variation over which the radii of gyration may be considered constant.

Relation (27a) now defines the limits on q in order that dynamic stability be possible: The requirement that $\bar{s}(2-\bar{s})$ be positive may be written as:

$$\text{(31a)} \quad 0 < \bar{s} < 2 \qquad \text{or} \qquad 0 < \frac{T_o + T_1 q}{H_o + H_1 q + H_2 q^2} < 1$$

For supersonic velocities and most bodies of revoluation, experience shows that for $\bar{s} = 0$ the center of mass is to the rear of the centroid. Beyond this point which is not more than two calibers away from the centroid $\bar{s} < 0$ and it is impossible to stabilize by spin. The situation can of course be improved by changing k_1. For some long bodies of revolution (fineness ratio greater than eight) $\bar{s} = 2$ when the center of mass is forward of the centroid. Forward of this point $\bar{s} > 2$ and it is impossible to stabilize by spin. This situation can be improved by changing k_2 and/or k_1.

In conclusion we will construct two representative stability plots for long bodies of revolution. The second plot is quite similar to those obtained for shorter models. Table I gives values of the aerodynamic coefficients and physical constants for a typical body and Figure 3 plots the square of the required spin against c.m. location for various values of α^*. ν^2 is used in order that $s = 1$ will appear as a straight line. A value of spin which is above an α^* curve corresponds to a missile with damping exponents greater than α.

TABLE I

$K_A = .01$	$K_D = .15$	$K_F = .3$	$K_H = 20$
$K_M = 4$	$K_N = 1$	$K_S = -9$	$K_T = -.3$
$k_1^{-2} = 8$	$k_2^{-2} = .2$	$\frac{\rho d^3}{m} = 5 \times 10^{-5}$	

Notice that there is only a small range of c.m. for which dynamic stability is possible. Point 0 is the optimum c.m. location and is about .4 cal rear of the centroid. If we now attempt to improve the stability characteristics of this hypothetical model by increasing k_2^{-2} from .2 to .3, we find that the situation is completely changed (see Fig 4.). The optimum point is .1 cal forward of the centroid and the region of possible dynamic stability is tremendously enlarged. Note, however, that the restriction on c.m. location reappears when damping of the order of $\alpha^* = 6$ is required.

STABILITY PLOT FOR $k_2^{-2} = 0.2$

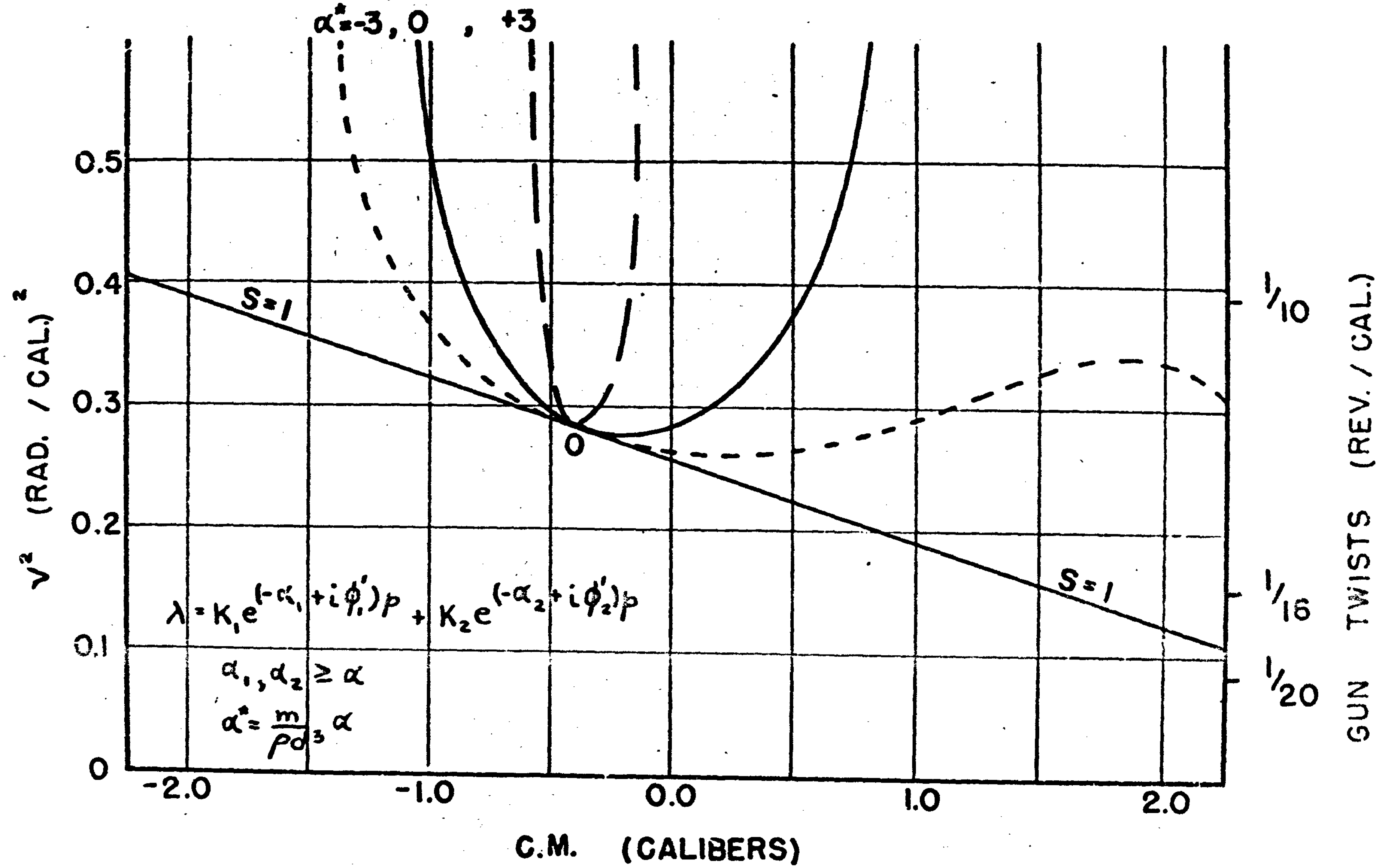

FIG. 3

STABILITY PLOT FOR $k_2^{-2} = 0.3$

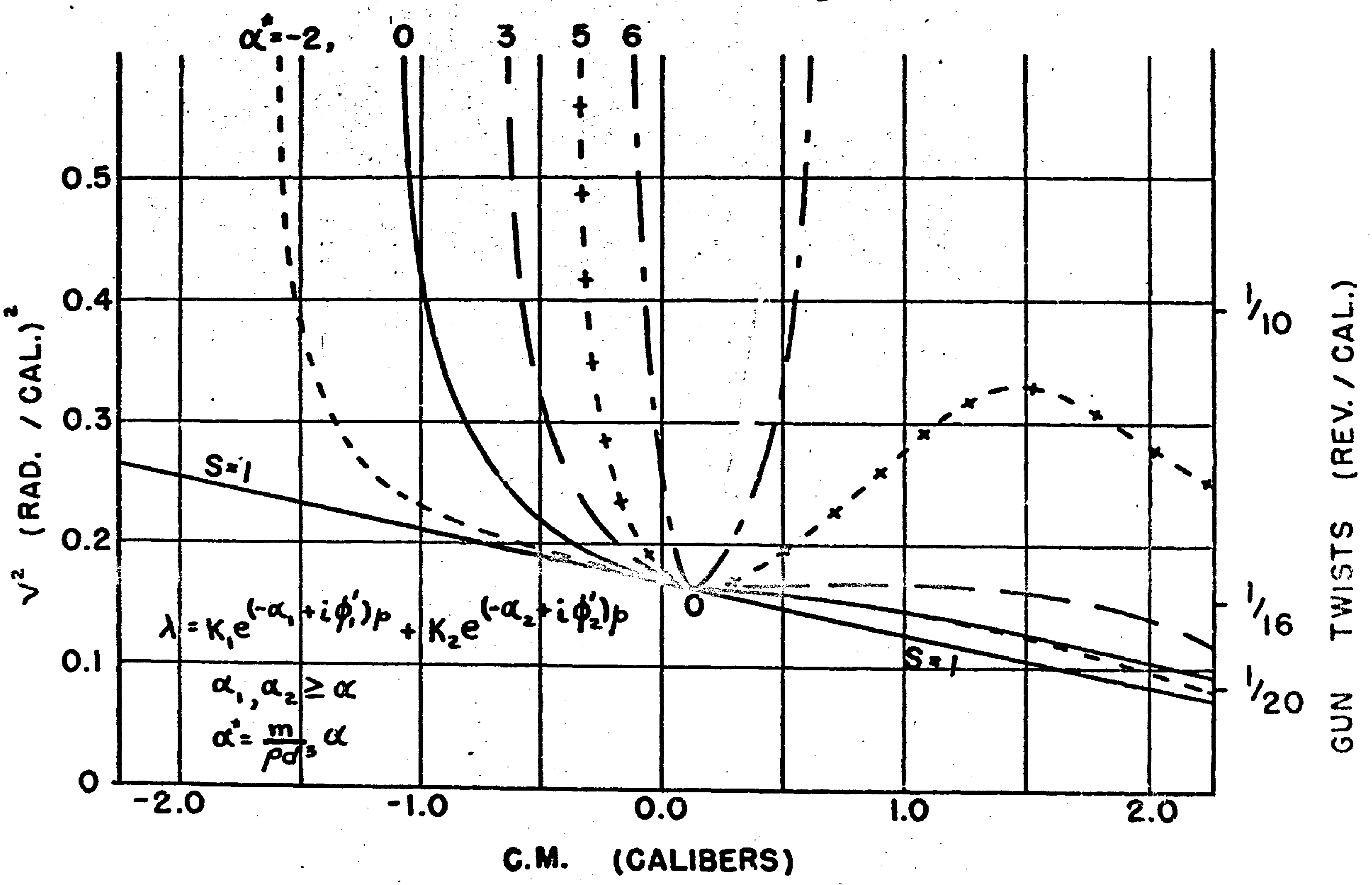

FIG. 4

ARMY---DG---ABERDEEN PROVING GROUND, MD.

The major effects that the consideration of the dynamic stability of statically unstable missiles has on the development of such spin stabilized missiles may be summarized in the following three statements:

(1) If there is no requirement on the size of damping and $3/4 < \bar{s} < 5/4$, the usual gyroscopic considerations apply ($s \geqq 1$).

(2) If there is no requirement on the size of damping and $\bar{s}$ is outside this interval, it must be improved either by a different mass distribution which varies k_1 and k_2 or by a different shape which changes the ballistic coefficients.

(3) If there is a minimum level of damping α, then $\bar{s}(\alpha)$ must be considered and the design procedure is that of (1) if $3/4 < \bar{s}(\alpha) < 5/4$ otherwise it is that of (2).

C. H. Murphy

C. H. MURPHY

TABLE OF SYMBOLS

A Axial moment of inertia

B Transverse moment of inertia

C $= \arctan \frac{b}{a}$

$D = J_D - \frac{md^2}{A} \cdot J_A$

$F = (F_1, F_2, F_3)$ aerodynamic force

$G = \gamma' - \left[\left(J_D - J_g - \frac{md^2}{B} J_H \right) + i\bar{v} \right] \gamma$

$H = (h_1, h_2, h_3)$ angular momentum vector or

$J_L - J_D + \frac{md^2}{B} \quad J_H$

H_0, H_1, H_2 coefficients in expansion of H + D as function of center of mass

$J_i = \frac{\rho d^3}{m} K_i; \; i \neq g$

$J_g = \frac{g_1 d}{u_1^2} \doteq - \frac{gd \sin \theta}{u^2}$

K_A Axial spin deceleration coefficient

K_D Drag coefficient

K_{DA} Axial drag coefficient

K_F Magnus[1] force due to yaw coefficient

K_H Damping moment coefficient

K_L Lift coefficient

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1. Terms in the cross force (A2) or cross moment (A4) which are zero for zero spin are called "Magnus" forces.

K_M Overturning (righting) moment coefficient

K_N Normal force coefficient

K_S Damping force coefficient

K_T Magnus moment coefficient

K_{XF} Magnus force due to cross angular velocity coefficient

K_{XT} Magnus moment due to cross angular velocity coefficient

K_1, K_2 Integration constants in (10)

$M = \frac{md^2}{B} J_M$

(M_1, M_2, M_3) Aerodynamic moment vector

$T = J_L - \frac{md^2}{A} J_T$

T_0, T_1 Coefficients in expansion of T as function of center of mass

$a = -\overline{m}$

$b = 2\,\overline{v}\,(2T-H-D)$

d diameter of missile

g acceleration due to gravity

k_1, k_2 radii of gyration

(g_1, g_2, g_3) vector acceleration due to gravity

(h_1, h_2, h_3) angular momentum vector

m mass of missile

$\overline{m} = \overline{v}^{\,2} - 4\,\frac{md^2}{B} J_M$

$p = \int_0^t \frac{u_1}{d}\,dt$ independent variable

$q = \lambda \exp 1/2 \int (H + J_g - i\overline{v})dp$ or shift in center of mass location

$r = 1/2 \left[(4M - \overline{v}^{\,2}) + 2\,i\overline{v}\,(2T - H - D) \right]^{1/2}$

s stability factor

$\bar{s}$ dynamic stability factor

s_1, s_2, s_3 symbols used in stability discussion of [5] and [6]

t time

u (u_1, u_2, u_3) vector velocity of center of mass

$w = (\ln q)'$

α minimum level of damping

α_1, α_2 damping of each epicycle arm

$\mathcal{E}, \mathcal{E}^*$ W.K.B. perturbation term

$\mathcal{E}_1 + i\mathcal{E}_2 = \bar{\mathcal{E}}(D - J_g)^{-1}$

$\bar{\mathcal{E}} = J_g(1 - \mathcal{E}_1) + D\mathcal{E}_1$

θ inclination of trajectory to horizontal plane

$\lambda = \frac{u_2}{u_1} + i\frac{u_3}{u_1}$ complex yaw

λ_R Yaw of repose

$\mu = \frac{\omega_2 d}{u_1} + i\frac{\omega_3 d}{u_1}$ complex angular velocity

$\nu = \frac{\omega_1 d}{u_1}$ spin in radians per caliber

$\bar{\nu} = \frac{A}{B}\nu$

ρ density of air

σ^2 symbol used in [5] and [6] see footnote to Equation (7)

$\phi_1', \phi_2', \phi_1'', \phi_2''$ turning rates and their derivatives for the epicycle arms

Ω angular velocity of coordinate system

$(\omega_1, \omega_2, \omega_3)$ angular velocity of missile

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APPENDIX A KELLEY-McSHANE FORCE SYSTEM

In order to describe the aerodynamic force acting on a missile in flight, we need first to set up a suitable coordinate system. This will consist of numbered axes so that the 1 axis lies along the missile's axis while the 2 and 3 axes are perpendicular to it. As a convenient shorthand we will consider the plane perpendicular to the missile's axis to be the complex plane with the 2 axis as the real axis and the 3 axis imaginary. Now if the velocity vector of the missile's center of mass is resolved along these axes, the yaw, λ, of the missile, which is the directed angle between the missile's axis and the velocity vector, can be expressed in complex form as $\lambda = \frac{u_2 + iu_3}{u_1}$. This expression is of course accurate for small yaw only.

The second and third components of the angular velocity vector $\omega = (\omega_1, \omega_2, \omega_3)$ are of importance in the consideration of the yawing motion. These can be non-dimensionalized and written in complex form as $\mu = \frac{\omega_2 d + i\,\omega_3 d}{u_1}$ where d is the diameter of the projectile. The aerodynamic problem can now be reduced to a dynamic problem by the assumption that the aerodynamic force and moment are functions of the air density ρ, its sound velocity a, axial velocity u_1, axial angular velocity ω_1, yaw λ, cross angular velocity μ, size of missile which is characterized by its diameter d and its shape. The Kelley-McShane theory makes the further simplification that the aerodynamic force and moment are linear functions of λ and μ. If these hypotheses are applied to a configuration possessing an angle of rotational symmetry less than 180° and a plane of mirror symmetry, the aerodynamic force and moment can be written in the following form [4], [6] [5]:

(A1) $F_1 = -\rho\, d^2\, u_1^2\, K_{DA}$

(A2) $F_2 + iF_3 = \rho\, d^2\, u_1^2 \left[\left(-K_N + i\nu K_F\right)\lambda + \left(\nu K_{XF} + iK_S\right)\mu\right]$

(A3) $M_1 = -\rho\, d^3\, u_1^2\, \nu\, K_A$

(A4) $M_2 + iM_3 = \rho\, d^3\, u_1^2 \left[\left(-\nu K_T - iK_M\right)\lambda + \left(-K_H + i\nu K_{XT}\right)\mu\right]$

where $\nu = \frac{\omega_1 d}{u_1}$

An important aspect of the above definitions is the dependence of the coefficients appearing in (A2) and (A4) on the location of the center of mass. In (A2) the aerodynamic force is independent of the location of the center of mass while the yaw λ is defined in terms of the velocity of the center of mass. The dependence of the moment coefficients in (A4) is more complex since they relate to the total aerodynamic moment about the center of mass and are also associated with λ.

Consider the case where the center of mass is shifted from the point 0 on the missile's axis to the point 0* which is located on the missile's axis at a distance of q calibers from 0. (Positive q will correspond to a shift toward the nose of the missile.) The vector $\overrightarrow{00^*}$ is in coordinate form (q d, 0, 0). All quantities relating to the configuration possessing the new center of mass will be marked by an asterisk.

Now if corresponding points of the two configurations possess the same motion, the total aerodynamic force on each configuration are identical and the total aerodynamic moment when computed about corresponding geometric points will be equal [3], [6], [26]. For the same motion we now compute the velocities of the points 0 and 0*.

(A5) $u^* = u + \omega \times (qd, 0, 0) = (u_1, u_2 + \omega_3 qd, u_3 - \omega_2 qd)$

Therefore

(A6) $u_1^* = u_1$

(A7) $\lambda^* = \lambda - iq\mu$

Since the angular velocity vector is independent of the location of the center of mass, we have

(A8) $\nu^* = \nu$

(A9) $\mu^* = \mu$

We can now express the force acting on the configuration with the new center of mass as

(A10) $F_1^* = -\rho\, d^2\, u_1^{*2}\, K_{DA}^{\ *} \doteq -\rho\, d^2\, u_1^{\,2}\, K_{DA}^{\ *}$

(A11) $F_2^* + iF_3^* = \rho d^2\, u_1^{*2} \left[\left(-K_N^{\ *} + i\nu^* K_F^{\ *}\right)\lambda^* + \left(\nu^* K_{XF}^{\ *} + iK_S^{\ *}\right)\mu^*\right]$

$$= \rho\, d^2\, u_1^{\,2} \left[\left(-K_N^{\ *} + i\nu K_F^{\ *}\right)\left(\lambda - iq\mu\right) + \left(\nu\, K_{XF}^{\ *} + iK_S^{\ *}\right)\mu\right]$$

If we equate the above with (A1) and (A2) and compare coefficients of λ and μ respectively we obtain the following:

(A12) $K_{DA}^* = K_{DA}$

(A13) $K_N^* = K_N$

(A14) $K_F^* = K_F$

(A15) $K_S^* = K_S - q\, K_N$

(A16) $K_{XF}^* = K_{XF} - q\, K_F$

Now in order to compare the aerodynamic moments we must first compute the moment for the second configuration about the point 0. This can be written as the moment about 0 in terms of the starred quantities:

(A17) $M^* + (qd, 0, 0) \times F^* = M^* - (qd, 0, 0) \times F = (M_1^*, M_2^* - F_3\, qd, M_3^* - F_3\, qd)$

Therefore

(A18) $M_1^* = -\rho\, u_1^2\, d^3\, K_A^*$

(A19) $(M_2^* - F_3\, qd) + i\,(M_3^* + F_2\, qd) = M_2^* + iM_3^* + iqd\,(F_2 + iF_3)$

$$= \rho d^3\, u_1^2 \Big\{ (-\nu K_T^* - iK_M^*)\,(\lambda - iq\mu) + (-K_H^* + i\nu K_{XT}^*)\,\mu$$

$$- q\,\Big[(-K_N + i\nu\, K_F)\,\lambda + (\nu\, K_{XF} + iK_S)\,\mu \Big] \Big\}$$

Equating (A18) and (A19) with (A3) and (A4) we have

(A20) $K_A^* = K_A$

(A21) $K_M^* = K_M - q\, K_N$

(A22) $K_T^* = K_T - q\, K_F$

(A23) $K_H^* = K_H - q\,(K_S + K_M) + q^2\, K_N$

(A24) $K_{XT}^* = K_{XT} - q\,(K_{XF} + K_T) + q^2\, K_F$

In order to obtain the form of (A1) and (A2) used in Appendix B we derive two final relations. Sometimes the force system is reduced to only two components which lie in the plane of yaw and are resolved along the trajectory and perpendicular to it. The first component is called the drag force D and the second is called the lift force L. They define two more coefficients by the relations

$$(A25) \quad D = -\rho\, d^2\, u_1^2\, K_D$$

$$(A26) \quad L = -\rho\, d^2\, u_1^2\, K_L\, \lambda$$

If we neglect Magnus terms and terms dependent on μ in (A2), we see that (A1) and (A2) describe the same type of force as (A25) and (A26) but differently directed. We now select the real axis so that it lies in the plane of yaw. Clearly (A25), (A26) are related to (A1) and (A2) by a rotation through the angle λ.

$$\therefore \quad (A25) \quad D = F_2 \sin\lambda + F_1 \cos\lambda$$

$$(A26) \quad L = F_2 \cos\lambda - F_1 \sin\lambda$$

Hence for small λ,

$$\therefore \quad (A27) \quad K_D = K_N\, \lambda \sin\lambda + K_{DA} \cos\lambda \doteq K_{DA}$$

$$(A28) \quad K_L = K_N \cos\lambda - K_{DA}\, \frac{\sin\lambda}{\lambda} \doteq K_N - K_D$$

If the discussion of the above paragraph seems to lack rigor, equation (A27) and (A28) may be taken as definitions of K_D and K_L. This can be done since only these relations and not their physical definitions are used in this report. Using these we can write (A1) and (A2) in their final form:

$$(A1') \quad F_1 = -\rho\, d^2\, u_1^2\, K_D$$

$$(A2') \quad F_2 + iF_3 = \rho\, d^2\, u_1^2 \left[(-K_L - K_D + \nu\, K_F)\lambda + (\nu K_{XF} + iK_S)\, \mu \right]$$

APPENDIX B: DERIVATION OF THE EQUATION OF YAWING MOTION

A right handed orthogonal coordinate system with axes numbered 1, 2, 3 moving with the missile and so orientated that the 1 axis always points toward the nose along the missile's axes will be used throughout this appendix. If we specify $\omega = (\omega_1, \omega_2, \omega_3)$ to be the angular velocity vector[1] of the missile in this coordinate system and take $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ to be the angular velocity of the coordinate system, the above restriction on the 1 axis provides the relations $\Omega_2 = \omega_2$ and $\Omega_3 = \omega_3$. Ω_1 is selected to be identically zero and the 2 axis is initially orientated to lie in the horizontal plane and point to the right. For the small yaw and small curvature of the trajectory which are assumed in this development, the 2 axis will always remain quite close to the horizontal plane. Our equations of motion are:

(B1) $$\frac{d}{dt}(m\,u) = (F_1, F_2, F_3) + m\,(g_1, g_2, g_3)$$

(B2) $$\frac{dH}{dt} = (M_1, M_2, M_3)$$

where

$u = (u_1, u_2, u_3)$ is the velocity vector of the center of gravity

$H = (h_1, h_2, h_3)$ is the angular momentum vector of the missile

(F_1, F_2, F_3) is the aerodynamic force vector acting on the missile

(g_1, g_2, g_3) is the acceleration due to gravity vector

(M_1, M_2, M_3) is the aerodynamic moment vector acting about the center of gravity

m is the mass of the missile

Since our coordinate system is not an inertia system, we have to

---

1. All linear and angular velocities are with respect to an inertia system fixed on the earth.

differentiate using the relation[2] $\frac{d}{dt} = \frac{d}{dt}_{\text{fixed axes}} + \Omega\, x.$

Although axis 1 is fixed in the missile the other two are moving with respect to it. This would make the angular momentum vector rather complicated were it not for the fortunate fact that the mass distribution of most missiles are effectively those of bodies of revolution. This implies that every direction normal to the axis of the missile is a principal axis of inertia and one transverse moment of inertia prevails. If A is the axial moment of inertia and B the transverse moment of inertia, then (h_1, h_2, h_3) is the vector $(A\omega_1, B\omega_2, B\omega_3)$.

It is convenient to make the plane normal to the missile's axis a complex plane with axis 2 the real axis and axis 3 the pure imaginary axis. For any vector equation this can be done by multiplying the third coordinate equation by i and adding to its second.

In the remaining equations of this section a dot will signify differentiation with respect to time.

With the above remarks in mind it is possible to obtain the following equations from (B1) and (B2).

(B3) $\dot{u}_1 + (\omega_2 u_3 - \omega_3 u_2) = \frac{F_1}{m} + g_1$

(B4) $(\dot{u}_2 + i\,\dot{u}_3) - i\,u_1\,(\omega_2 + i\,\omega_3) = \frac{F_2 + i\,F_3}{m} + (g_2 + i\,g_3)$

(B5) $\dot{\omega}_1 = \frac{M_1}{A}$

(B6) $(\dot{\omega}_2 + i\,\dot{\omega}_3) - i\,\omega_1\,(\omega_2 + i\,\omega_3)\,\frac{A}{B} = \frac{M_2 + i\,M_3}{B}$

2. If we take X_1, X_2, X_3 to be a triad of unit vectors pointing along the respective axes of the coordinate system, any vector V can be written as $V = V_1X_1 + V_2X_2 + V_3X_3$. Differentiating with respect to time using the dot convention this yields $\dot{V} = (\dot{V}_1X_1 + \dot{V}_2X_2 + \dot{V}_3X_3) + (V_1\dot{X}_1 + V_2\dot{X}_2 + V_3\dot{X}_3)$. The first term is $\left(\frac{d}{dt}\right)_{\text{fixed axes}}$ and the second can be written as $\Omega \times V$. This is the definition for the angular velocity vector of the coordinate system. In component form Ω is $(\dot{X}_2 \cdot X_3)\, X_1 + (\dot{X}_3 \cdot X_1)X_2 + (\dot{X}_1 \cdot X_2)\, X_3$.

The equations are now made non-dimensional by use of the axial velocity u_1 and the missile's diameter d. This introduces the non-dimensional dependent variables:

$$\nu = \frac{\omega_1 d}{u_1} \quad \text{(spin in radians per caliber)}$$

$$\lambda = \frac{u_2 + i\, u_3}{u_1} \quad \text{(complex yaw)}$$

$$\mu = \frac{\omega_2 d + i\, \omega_3 d}{u_1} \quad \text{(complex angular velocity)}$$

The independent variable time is replaced by $p = \int_0^t \frac{u_1 \, dt}{d}$ which for small yaw is approximately arc length measured along the trajectory in calibers. If a prime is introduced to represent differentiation with respect to p, we have the relation $(\ )^{\bullet} = (\ )' \frac{u_1}{d}$. From equations (B3) - (B6) we can now write:

$$\text{(B7)} \quad \frac{u_1'}{u_1} + \frac{(\omega_2 u_3 - u_2 \omega_3) d}{u_1^2} = \frac{F_1 d}{m u_1^2} + \frac{g_1 d}{u_1^2}$$

$$\text{(B8)} \quad \lambda' + \frac{u_1'}{u_1} \lambda - i\mu = \frac{(F_2 + i F_3) d}{m u_1^2} + \frac{(g_2 + i g_3) d}{u_1^2}$$

$$\text{(B9)} \quad \nu' + \frac{u_1'}{u_1} \nu = \frac{M_1 d^2}{A u_1^2}$$

$$\text{(B10)} \quad \mu' + \frac{u_1'}{u_1} \mu - i \frac{A}{B} \nu\mu = \frac{(M_2 + i M_3) d^2}{B u_1^2}$$

From equations (A1'), (A2'), (A3) and (A4) the linearized force system can be written as:

$$\text{(B11)} \quad F_1 = \frac{-m u_1^2}{d} J_D$$

$$\text{(B12)} \quad F_2 + i F_3 = \frac{m u_1^2}{d} \left[(-J_L - J_D + i\nu J_F)\lambda + (\nu J_{XF} + i J_S)\mu \right]$$

$$\text{(B13)} \quad M_1 = -m u_1^2 \nu J_A$$

(B14) $M_2 + i\,M_3 = m\,u_1^2 \left[(-\nu\, J_T - i\, J_M)\,\lambda + (-J_H + i\,\nu\, J_{XT})\mu\right]$

where $J = \frac{\rho\, d^3}{m} K$ for all subscripts, ρ is the air density, and the K's are the aerodynamic coefficients. (See Table of Symbols and Coefficients)

Using (B11) we can now write for (B7)

(B15) $$\frac{u_1'}{u_1} = -\, J_D + J_g - \frac{(\omega_2 u_3 - \omega_3 u_2)\, d}{u_1^2}$$

where $J_g = \frac{g_1 d}{u_1^2}$.

For the small yaw and transverse angular velocity required by the linearity assumption the third term on the right side of (B15) can be neglected in comparison with $-J_D + J_g$

$\therefore$ (B16) $$\frac{u_1'}{u_1} = -\, J_D + J_g$$

Using (B16) and definitions (B12) - (B14) equations (B8) - (B10) may be rewritten as:[1]

(B17) $\lambda' - i\,\mu = (-\,J_L + i\,\nu\, J_F)\,\lambda + (\nu\, J_{XF} + i\, J_3)\mu + \gamma$

(B18) $\nu' = (D - J_g)\,\nu$

(B19) $\mu' - (J_D - J_g + i\,\frac{A}{B}\,\nu)\,\mu = k_2^{-2}\left[(-\,\nu\, J_T - i\, J_M)\lambda + (-J_H + i\nu\, J_{XT})\mu\right]$

where $$\gamma = \frac{(g_2 + i\, g_3)d}{u_1^2} - J_g\,\lambda = \frac{\left[(g_2 - g_1\lambda_2) + i\,(g_3 - g_1\,\lambda_3)\right] d}{u_1^2}$$

$D = J_D - k_1^{-2}\, J_A$

$k_1^{-2} = \frac{m\, d^2}{A}$ (k_1 is axial radius of gyration in calibers)

$k_2^{-2} = \frac{m\, d^2}{B}$ (k_2 is transverse radius of gyration in calibers)

---

1. The grouping of $J_g \lambda$ in the γ was a correction introduced by Professor McShane in order to make the treatment of gravity more accurate. A more extended discussion of this may be found in [2].

We now operate on (B17) with the operator $\left[\frac{d}{dp} + (k_2^{-2} J_H - J_D + J_g) - i \nu (\frac{A}{B} + k_2^{-2} J_{XT})\right]$, multiply (B19) by $\nu J_{XF} + i (1 + J_S)$, and add. Assuming that derivatives of force coefficients can be neglected and using (B18) to eliminate ν', the result reduces to:

$$\text{(B20)} \quad \lambda'' + \left[H + J_g - i\bar{\nu} X\right]\lambda' + (-M - i\bar{\nu} T)\lambda = G + \nu J_{XF}(D - J_g)\mu$$

where

$$H = J_L - J_D + k_2^{-2} J_H$$

$$\bar{\nu} = \frac{A}{B}\nu$$

$$X = 1 + k_1^{-2} J_{XT} + \frac{B}{A} J_F$$

$$M = k_2^{-2}\Big[\left(J_M + \nu^2 k_1^{+2} J_F\right) + \nu^2 (J_F J_{XT} - J_T J_{XF})$$
$$+ (J_M J_S - J_L J_H)\Big] + J_L (J_D - J_g)$$

$$T = J_L - k_1^{-2} J_T + k_1^{-2}\left[J_F J_H - J_{XF} J_M - J_S J_T + J_L J_{XT} - \frac{B}{A} J_F J_A\right]$$

$$G = \gamma' - \left[(J_D - k_2^{-2} J_H - J_g) + i\bar{\nu}(1 + k_1^{-2} J_{XT})\right]\gamma$$

The upper case letters with the exception of G are selected in order to identify the moment coefficient which is the principal constituent. The quite formidable expressions above can be simplified by certain quite reasonable size assumptions. We assume that $|J_F| < 10^{-4}$, $|J_i| < 3 \times 10^{-3}$ otherwise, $1 < \frac{B}{A} < 10^2$, $k_1^{-2} < 10$, $k_2^{-2} \leq 2$, $|\nu| < 1$, and that $|\lambda| < 1$. Since $\frac{\rho d^3}{m}$ is usually about 5×10^{-5}, this restricts the magnitude of K_F to less than 2 and that of other K_i's to less than 60. The requirement for the special case of J_g reduces to $d^{1/2} < (10^{-2}u)$ where d is in feet and u in feet/sec. From (B17) it can be seen that μ is comparable with $i\lambda'$ and hence the second term on right side of (B20) can be neglected in comparison with $i\nu X \lambda'$. X can clearly be very well approximated by 1 and similar approximations apply to the other terms. There results the following good approximation of (B20).

$$\text{(B21)} \quad \lambda'' + (H + J_g - i\bar{\nu})\lambda' + (-M - \bar{\nu} T)\lambda = G$$

where

$$H = J_L - J_D + k_2^{-2} J_H$$

$$M = k_2^{-2} (J_M + \nu^2 k_1^2 J_F)$$

$$T = J_L - k_1^{-2} J_T$$

$$G = \gamma' - \left[(J_D - k_2^{-2} J_H - J_g) + i\bar{\nu}\right]\gamma$$

Note that the form of (B21) differs from (B20) only by the absence of X and $\nu J_{XF}(D-J_g)\mu$. Since the general stability analysis of the report is stated in terms of M, H, and T, all the assumptions of (B21) except $X = 1$ and $\nu J_{XF}(D-J_g)\mu = 0$ may be avoided by inserting the values of H, M and T given after equation (B20).

An important case where the above assumptions do not apply is that of the airship. Here $\frac{\rho d^3}{m}$ is of the order of one. If the effect of drag and gravity are neglected and spin is taken to be zero equation (B20) reduces to:

(B22) $\lambda'' + H\lambda' - M\lambda = 0$

where $H = J_L + k_2^{-2} J_H$

$$M = k_2^{-2} J_M(1 + J_S) - J_L J_H$$

Since for this configuration J_H and J_L are positive and J_S is usually negative and less than one, we see that M can be negative even when K_M is positive. This provides the interesting result that a statically unstable configuration can be dynamically stable without spin[1]. This is, of course, limited to configurations of small density.

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1. An explicit example of this is given in Durand, Aerodynamic Theory, Vol. VI, pages 110 - 112. The neglection of drag and gravity is quite valid since drag is usually neutralized by the ship's propulsion system and gravity by the ship's buoyancy. The assumptions that the center of buoyancy is located at the center of mass and that apparent mass of the air due to flow around the ship can be neglected are, however, implicitly implied in (B22). Since the center of buoyancy is usually located above the center of mass and the additional consideration of apparent mass affects M by not more than 10%, (B22) is still reasonably valid for the horizontal component of yaw, λ_H. Actually only the behavior of this component is considered in Durand.

APPENDIX C: CONVERSION FROM BALLISTIC TO AERODYNAMIC NOMENCLATURE

The work of this report has been done in terms of the ballistic K's which are little known outside the field of ballistics and may be quite confusing to an aerodynamicist who does his dynamic stability analyses in terms of the aerodynamic C's. It is therefore quite worthwhile to express the results of this report in terms of these symbols. This effort is handicapped however, by the three facts:

1. The missiles usually treated in ballistics have a rotation symmetry which results in pairs of aerodynamic coefficients being equal and hence corresponding to only one ballistic coefficient.

2. In ballistics the missiles usually have a high rate of spin and Magnus effects have to be considered to which there are no corresponding aerodynamic coefficients.

3. Terms involving the rate of change of angle of attack appear in most aerodynamic stability analysis while no such terms appear in the usual ballistic force system.

The axial components of the aerodynamic force and moment are usually defined in aerodynamic nomenclature as:

$$\begin{aligned} X &= -1/2\,\rho\,V^2\,S\,C_D \\ L &= 1/2\,\rho\,V^2\,S\,b\left(\frac{pb}{2V}\right)C_{\ell_p} \end{aligned} \tag{C1}$$

From this we see that

$$\begin{aligned} K_D &= 1/2\,S/d^2\,C_D \\ K_A &= -1/4\,(Sb^2/d^4)\,C_{\ell_p} \\ \nu &= \frac{2d}{b}\,(pb/2V) \end{aligned} \tag{C2}$$

If the transverse components of the aerodynamic force and moment are assumed to be linear functions of yaw, change in yaw, and angular velocity, and Magnus coupling is introduced, we have the following definitions:

$$Y = (1/2\,\rho\,V^2\,S)\left\{\left[C_{Y_\beta}\,\beta + C_{Y_r}\left(\frac{br}{2V}\right) + C_{Y_{\dot\beta}}\left(\frac{b\dot\beta}{2V}\right)\right] + \left[C_{Y_{p\alpha}}\,\alpha + C_{Y_{pq}}\left(\frac{cq}{2V}\right) + C_{Y_{p\dot\alpha}}\left(\frac{c\dot\alpha}{2V}\right)\right]\left(\frac{pb}{2V}\right)\right\}$$

$$Z = (1/2)\ \rho\ V^2\ S)\left\{\left[C_{Z_\alpha}\ \alpha + C_{Z_q}\ \left(\frac{cq}{2V}\right) + C_{Z_{\dot\alpha}}\ \left(\frac{c\dot\alpha}{2V}\right)\right] + \left[C_{Z_{p\beta}}\ \beta + C_{Z_{pr}}\left(\frac{br}{2V}\right) + C_{Z_{p\dot\beta}}\left(\frac{b\dot\beta}{2V}\right)\right]\left(\frac{pb}{2V}\right)\right\}$$

(C3)

$$M = (1/2\ \rho\ V^2\ S\ c)\left\{\left[C_{m_\alpha}\ \alpha + C_{m_q}\left(\frac{cq}{2V}\right) + C_{m_{\dot\alpha}}\left(\frac{c\dot\alpha}{2V}\right)\right] + \left[C_{m_{p\beta}}\ \beta + C_{m_{pr}}\ \left(\frac{br}{2V}\right) + C_{m_{p\dot\beta}}\left(\frac{b\dot\beta}{2V}\right)\right]\left(\frac{pb}{2V}\right)\right\}$$

$$N = (1/2\ \rho\ V^2\ S\ b)\left\{\left[C_{n_\beta}\ \beta + C_{n_r}\ \left(\frac{br}{2V}\right) + C_{n_{\dot\beta}}\ \left(\frac{b\dot\beta}{2V}\right)\right] + \left[C_{n_{p\alpha}}\ \alpha + C_{n_{pq}}\left(\frac{cq}{2V}\right) + C_{n_{p\dot\alpha}}\left(\frac{c\dot\alpha}{2V}\right)\right]\left(\frac{pb}{2V}\right)\right\}$$

If the missile is assumed to possess trigonal or greater rotational symmetry, it follows that [4], [6], [25], [27]:

$$C_{Y_\beta} = C_{Z_\alpha} \equiv C_{N_\alpha} \qquad -C_{Y_{p\alpha}} = C_{Z_{p\beta}} \equiv C_{N_{p\alpha}}$$

$$-C_{Y_r}\frac{b}{c} = C_{Z_q} \equiv C_{N_q} \qquad C_{Y_{pq}} = C_{Z_{pr}}\frac{b}{c} \equiv C_{N_{pq}}$$

$$C_{Y_{\dot\beta}}\frac{b}{c} = C_{Z_{\dot\alpha}} \equiv C_{N_{\dot\alpha}} \qquad C_{Y_{p\dot\alpha}} = C_{Z_{p\dot\beta}}\left(\frac{b}{c}\right) \equiv C_{N_{p\dot\alpha}}$$

(C4)

$$-C_{m_\alpha} = C_{n_\beta}\left(\frac{b}{c}\right) \equiv -C_{M_\alpha} \qquad C_{m_{p\beta}} = C_{n_{p\alpha}}\left(\frac{b}{c}\right) \equiv C_{M_{p\alpha}}$$

$$C_{m_q} = C_{n_r}\left(\frac{b}{c}\right)^2 \equiv C_{M_q} \qquad -C_{m_{pr}}\left(\frac{b}{c}\right) = C_{n_{pq}}\left(\frac{b}{c}\right) \equiv -C_{M_{pq}}$$

$$-C_{m_{\dot\alpha}} = C_{n_{\dot\beta}}\left(\frac{b}{c}\right)^2 \equiv -C_{M_{\dot\alpha}} \qquad C_{m_{p\dot\beta}}\left(\frac{b}{c}\right) = C_{n_{p\dot\alpha}}\left(\frac{b}{c}\right) \equiv C_{M_{p\dot\alpha}}$$

The third set of symbols is introduced in order to emphasize the existence of symmetry and will be employed throughout the remainder of this appendix. If we insert these symbols into (C3), multiply the second and fourth equations by i and add to the first and third respectively there results:

$$\text{(C5)}\qquad \begin{aligned} Y + iZ = (1/2\,\rho\, V^2\, S)\Big\{&\Big[C_{N_\alpha} + i\left(\tfrac{pb}{2V}\right) C_{N_{p\alpha}}\Big](\beta + i\alpha) \\ &+ \Big[\left(\tfrac{pb}{2V}\right) C_{N_{pq}} + i\, C_{N_q}\Big]\left(\frac{cq + i\,cr}{2V}\right) \\ &+ \Big[C_{N_{\dot\alpha}} + i\,\tfrac{pb}{2V}\, C_{N_{p\dot\alpha}}\Big]\left(\frac{c\dot\beta + i\,c\dot\alpha}{2V}\right)\Big\} \\ M + iN = (1/2\,\rho\, V^2\, cS)\Big\{&\Big[\left(\tfrac{pb}{2V}\right) C_{M_{p\alpha}} - i\, C_{M_\alpha}\Big](\beta + i\alpha) \\ &+ \Big[C_{M_q} - i\left(\tfrac{pb}{2V}\right) C_{M_{pq}}\Big]\left(\frac{cq + i\,cr}{2V}\right) \\ &+ \Big[\left(\tfrac{pb}{2V}\right) C_{M_{p\dot\alpha}} - i\, C_{M_{\dot\alpha}}\Big]\left(\frac{c\,\dot\beta + i\,c\dot\alpha}{2V}\right)\Big\} \end{aligned}$$

If equation (C5) is compared with equations (A2) and (A4), the Magnus and non-Magnus static coefficients are easily related.

$$\text{(C6)}\qquad \begin{aligned} K_N &= -\,1/2\; S/d^2\; C_{N_\alpha} \\ K_M &= 1/2\; Sc/d^3\; C_{M_\alpha} \\ K_F &= 1/4\; Sb/d^3\; C_{N_{p\alpha}} \\ K_T &= -\,1/4\; Scb/d^4\; C_{M_{p\alpha}} \end{aligned}$$

The relationships between the remaining dynamic coefficients are somewhat more complicated. Fortunately it can easily be shown that the remaining Magnus coefficients are lost in the differential equations of yawing motion due to the J^2 connection. It therefore, remains only to connect two ballistic coefficients, K_S and K_H, with

four aerodynamic coefficients, C_{N_q}, $C_{N_{\dot\alpha}}$, C_{M_q} and $C_{M_{\dot\alpha}}$

In order to do this we need only to consider the purpose of this work, namely to state the results of this report in aerodynamic nomenclature. Since this report is concerned with stability, the only contribution of the aerodynamic coefficients is how they appear in the basic differential equations. This means that in order to obtain the partner of K_H we see what coefficient appears in the corresponding point of the differential equation similar to (1) which is based on the aerodynamic force system (see [27] for example). By this tactic we have:

$$\text{(C7)} \qquad K_H \rightarrow -1/4 \, \frac{c^2 S}{d^4} \left(C_{M_q} + C_{M_{\dot\alpha}} \right)$$

Since the major function of K_S is its contributions to K_H when the center of mass is altered we have:

$$\text{(C8)} \qquad K_S \rightarrow 1/4 \, \frac{cS}{d^3} \left(C_{N_q} + C_{N_{\dot\alpha}} \right)$$

Note: The method of obtaining (C7) and (C8) is not too desirable. It would, of course, be more satisfying to enlarge the ballistic force system so that there would exist a one-to-one correspondence. (Efforts are being made in this direction at the present time.) It also should be noted that (C7) follows from a comparison of the homogeneous equations. In the yaw of repose, equation (11), K_H should be replaced by $-1/4 \, \frac{c^2 S}{d^4} C_{M_q}$

By use of (C2), (C6), (C7) and (C8) it is now possible to convert our symbols. We will merely tabulate the results.[1] (K_L will be replaced $K_N - K_D$ for this purpose.)

$$H = \frac{-\rho dS}{2m} \left[C_{N_\alpha} + 2\, C_D + 1/2 \, k_2^{-2} \left(\frac{c}{d}\right)^2 \left(C_{M_q} + C_{M_{\dot\alpha}} \right) \right]$$

$$\bar{\nu} = \frac{2d}{b} \left(\frac{pd}{2V}\right) \frac{A}{B}$$

$$M = k_2^{-2} \, \frac{\rho c S}{2m} \, C_{M_\alpha}$$

$$T = \frac{-\rho dS}{2m} \left[C_{N_\alpha} + C_D - k_1^{-2} \left(\frac{cb}{2d^2}\right) C_{M_{p\alpha}} \right]$$

1. In order to avoid confusion α^* in $\bar{s}(\alpha^*)$ will be replaced by γ^*.

$$D = \frac{\rho d S}{2m}\left[C_D + k_1^{-2}\ \frac{b^2}{2d^2}\ C_{\ell_p}\right]$$

$$s = \frac{A^2\ p^2}{4B\ (1/2\ V^2\ \rho S\ c\ C_{M_\alpha})}$$

$$\bar{s}(\gamma^*) = \frac{2\left[C_{N_\alpha} + C_D - k_1^{-2}\ (\frac{cb}{2d^2})\ C_{M_{p\alpha}}\right] + \gamma^*}{C_{N_\alpha} + C_D + 1/2\ k_2^{-2}\ (\frac{c}{d})^2\ (C_{M_q} + C_{M_{\dot\alpha}}) - k_1^{-2}\ \frac{b^2}{2d^2}\ C_{\ell_p} + \gamma^*}$$

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